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Global heat kernel estimate for relativistic stable processes in exterior open sets

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Abstract

In this paper, sharp two-sided estimates for the transition densities of relativistic α-stable processes with mass $m \in (0, 1]$ in $C^{1,1}$ exterior open sets are established for all time $t > 0$. These transition densities are also the Dirichlet heat kernels of $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$ with $m \in (0, 1]$ in $C^{1,1}$ exterior open sets. The estimates are uniform in m in the sense that the constants are independent of $m \in (0, 1]$. As a corollary of our main result, we establish sharp two-sided Green function estimates for relativistic α-stable processes with mass $m \in (0, 1]$ in $C^{1,1}$ exterior open sets.

Keywords: Symmetric α-stable process; Relativistic stable process; Heat kernel; Transition density; Green function; Exit time; Lévy system; Parabolic Harnack inequality

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1. Introduction

Let $d \geqslant 1$ and $\alpha \in (0,2)$. For any $m \geqslant 0$, a relativistic α-stable process X^m in $\mathbb{R}^d$ with mass m is a Lévy process with characteristic function given by

$$\mathbb{E}\big[\exp\big(i\xi \cdot \big(X_t^m - X_0^m\big)\big)\big] = \exp\big(-t\big(\big(|\xi|^2 + m^{2/\alpha}\big)^{\alpha/2} - m\big)\big), \quad \xi \in \mathbb{R}^d. \tag{1.1}$$

When $m = 0$, X^m is simply a (rotationally) symmetric α-stable process in $\mathbb{R}^d$. The infinitesimal generator of X^m is $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$. When $\alpha = 1$, the infinitesimal generator reduces to the free relativistic Hamiltonian $m - \sqrt{-\Delta + m^2}$. There exists a huge literature on the properties of relativistic Hamiltonians (for example, see [3,16,18,22,23]). Relativistic α-stable processes have been studied recently in [13,14,17,19–21,25].

Recall that an open set D in $\mathbb{R}^d$ (when $d \geqslant 2$) is said to be a (global) $C^{1,1}$ open set if there exist a localization radius $r_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \ldots, 0)$, $\|\nabla\phi\|_\infty \leqslant \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \leqslant \Lambda_0|x - z|$, and an orthonormal coordinate system $y = (y_1, \ldots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ such that $B(z, r_0) \cap D = B(z, r_0) \cap \{y\colon\ y_d > \phi(\widetilde{y})\}$. We call the pair (r_0, Λ_0) the characteristics of the $C^{1,1}$ open set D. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set can be unbounded and disconnected.

For an open set $D \subset \mathbb{R}^d$, let $X^{m,D}$ be the subprocess of X^m killed upon exiting D. It is easy to see (cf. [13]) that $X^{m,D}$ has a jointly continuous transition density function $p_D^m(t, x, y)$ with respect to the Lebesgue measure on D. p_D^m is also called the Dirichlet heat kernel of $m - (-\Delta + m^{2/\alpha})^{\alpha/2}|_D$ with zero exterior condition.

A relativistic α-stable process is a discontinuous Markov process. Sharp estimates on the transition density functions of discontinuous Markov processes are of current research interests (see [1,4–7,12,13] and the reference therein). Dirichlet heat kernel estimates for symmetric stable processes were first obtained in [8] on $C^{1,1}$ open sets for $t \leqslant 1$ and for all $t > 0$ when the $C^{1,1}$ open set is bounded. In [2], Dirichlet heat kernel estimates for symmetric stable processes were obtained for a large class of non-smooth open sets in terms of surviving probabilities. In [15], global Dirichlet heat kernel estimates for symmetric stable processes are derived for $C^{1,1}$ exterior open sets as well as for half-space-like open sets. The ideas of [8] have been adapted to establish sharp two-sided estimates for the Dirichlet heat kernels of other discontinuous Markov processes in open sets, see [9–11]. In particular, the following result is established in [10, Theorem 1.1]. In this paper, for any $a, b \in \mathbb{R}$, we use the notations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Theorem 1.1. *Suppose that $\alpha \in (0,2)$ and D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (r_0, Λ_0). Let $\delta_D(x)$ be the Euclidean distance between x and D^c.*

(i) *For any $M > 0$ and $T > 0$, there exists $c_1 = c_1(d, \alpha, r_0, \Lambda_0, M, T) > 1$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$\frac{1}{c_1}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right)$$

$$\leqslant p_D^m(t,x,y)$$
$$\leqslant c_1\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/(16))}{|x-y|^{d+\alpha}}\right), \qquad (1.2)$$

where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$.

(ii) *Suppose in addition that D is bounded. For any $M > 0$ and $T > 0$, there exists $c_2 = c_2(d, \alpha, r_0, \Lambda_0, M, T, \mathrm{diam}(D)) > 1$ such that for any $m \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,*

$$c_2^{-1} e^{-t\lambda_1^{\alpha,m,D}} \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} \leqslant p_D^m(t,x,y) \leqslant c_2 e^{-t\lambda_1^{\alpha,m,D}} \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},$$

where $\lambda_1^{\alpha,m,D} > 0$ is the smallest eigenvalue of the restriction of $(m^{2/\alpha} - \Delta)^{\alpha/2} - m$ in D with zero exterior condition.

Note that, although the small time estimates on $p_D^m(t,x,y)$ in Theorem 1.1(i) are valid for all $C^{1,1}$ open sets, the large time estimates in Theorem 1.1(ii) are only for bounded $C^{1,1}$ open sets. As one sees from the case of symmetric α-stable processes in [15], the large time heat kernel estimates for unbounded open sets are typically very different from that in the bounded open sets and depend on the geometry of the unbounded open sets. Sharp two-sided estimates on $p_D^m(t,x,y)$ valid for all time $t > 0$ have recently been established for half-space-like $C^{1,1}$ open sets in [11] by using some ideas from [15]. The goal of this paper is to establish sharp two-sided estimates on $p_D^m(t,x,y)$ for exterior $C^{1,1}$ open sets that hold for all $t > 0$.

Recall that an open set D in $\mathbb{R}^d$ is called an exterior open set if D^c is compact. For any $m, b, c > 0$, we define a function $\Psi_{d,\alpha,m,b,c}(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$\Psi_{d,\alpha,m,b,c}(t,x,y) := \begin{cases} t^{-d/\alpha} \wedge \frac{t\phi(c^{-1}m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} & \text{when } t \in (0, b/m], \\ m^{d/\alpha-d/2} t^{-d/2} \exp(-c^{-1}(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t})) & \text{when } t \in (b/m, \infty), \end{cases} \qquad (1.3)$$

where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$. The following is the main result of this paper.

Theorem 1.2. *Suppose that $\alpha \in (0,2)$, $d \geqslant 3$, $M > 0$, $b > 0$, $R > 0$ and D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (r_0, Λ_0) and $D^c \subset B(0,R)$. Then there are constants $c_i = c_i(d, \alpha, M, b, r_0, \Lambda_0, R) > 1$, $i = 1, 2$, such that for every $m \in (0, M]$, $t > 0$ and $(x, y) \in D \times D$,*

$$p_D^m(t,x,y) \leqslant c_1\left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,b,c_2}(t,x,y)$$

and

$$p_D^m(t,x,y) \geqslant c_1^{-1}\left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,b,1/c_2}(t,x,y).$$

It is known (see Theorem 2.1 below) that there are constants $c_3 > 1$ and $c_4 \geqslant 1$ such that for all $m > 0$ and $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_3^{-1} \Psi_{d,\alpha,m,1,1/c_4}(t, x, y) \leqslant p^m(t, x, y) \leqslant c_3 \Psi_{d,\alpha,m,1,c_4}(t, x, y). \tag{1.4}$$

By integrating the sharp heat kernel estimates in Theorem 1.2 (with $b = 1$) over $y \in D$ and using (1.4), one can easily conclude that there is a constant $c_5 = c_5(d, \alpha, M, r_0, \Lambda_0, R) \geqslant 1$ so that for every $m \in (0, M]$, $x \in D$ and $t > 0$,

$$c_5^{-1} \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \leqslant \mathbb{P}_x\big(\tau_D^m > t\big) \leqslant c_5 \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2},$$

where $\tau_D^m = \inf\{t > 0\colon\ X_t^m \notin D\}$. We emphasize that the sharp heat kernel estimates in Theorem 1.2 hold uniformly in $m \in (0, M]$. Thus passing $m \downarrow 0$ recovers the sharp heat kernel estimates for symmetric α-stable processes in exterior $C^{1,1}$ open sets when dimension $d \geqslant 3$ that were previously obtained in [15]. (The estimates in [15] hold for every $d \geqslant 2$.) The large time upper bound estimate in Theorem 1.2 is quite easy to establish, which is given at the end of Section 2. The main task of this paper is to establish the large time lower bound estimate for $p_D^m(t, x, y)$. Comparing with the case of symmetric stable processes, due to the fact that the Lévy densities of relativistic stable processes decay exponentially fast at infinity, the large time lower bound estimates for p_D^m is much harder to establish. The reason that we assume $d \geqslant 3$ in Theorem 1.2 is that, due to Chung–Fuck's recurrence criterion for Lévy processes, relativistic stable processes are transient if and only if $d \geqslant 3$.

Integrating the heat kernel estimates in Theorem 1.2 in $t \in (0, \infty)$, one gets the following sharp two-sided Green function estimates of X^m in exterior $C^{1,1}$ open sets, which is uniform in $m \in (0, M]$.

Theorem 1.3. *Suppose that $d \geqslant 3$, $M > 0$, $R > 0$ and D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (r_0, Λ_0) and $D^c \subset B(0, R)$. Then there is a constant $c = c(d, \alpha, M, r_0, \Lambda_0, R) > 1$ such that for every $m \in (0, M]$ and $(x, y) \in D \times D$,*

$$\begin{aligned} &c^{-1} \frac{1 + (m^{1/\alpha}|x-y|)^{2-\alpha}}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1}\right)^{\alpha/2} \\ &\quad \leqslant G_D^m(x, y) \leqslant c \frac{1 + (m^{1/\alpha}|x-y|)^{2-\alpha}}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1}\right)^{\alpha/2}. \end{aligned}$$

Taking $m \downarrow 0$, the estimates in Theorem 1.3 recover the sharp Green function estimates for symmetric α-stable processes in exterior $C^{1,1}$ open sets when $d \geqslant 3$ that was previously established in [15] for any dimension $d \geqslant 2$.

The rest of the paper is organized as follows. In Section 2, we summarize some basic properties of relativistic stable processes and give the proof of the upper bound estimate in Theorem 1.2. In Section 3, we present interior lower bound estimates for p_D^m in exterior open sets. Lower bound estimates for $p_D^m(t, x, y)$ up to the boundary are established in Section 4 for $t \leqslant T/m$ and in Section 5 for $t > T/m$. The proof of Theorem 1.3 is given in Section 6.

Throughout this paper, we assume that $\alpha \in (0, 2)$ and $m > 0$. The values of the constants C_1, C_2, C_3 will remain the same throughout this paper, while $c_1, c_2, \ldots$ stand for constants whose values are unimportant and which may change from location to location. The labeling of the constants $c_1, c_2, \ldots$ starts anew in the proof of each result. The dependence of the constant c on the dimension d will not be mentioned explicitly. We will use ":=" to denote a definition, which is read as "is defined to be". We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure and $aA := \{ay\colon\ y \in A\}$ for $a > 0$. For two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1, c_2 so that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ in the common domain of definitions for f and g.

2. Basic properties of relativistic stable processes

A symmetric α-stable process $X = \{X_t,\ t \geqslant 0,\ \mathbb{P}_x,\ x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$, where $d \geqslant 1$, is a Lévy process whose characteristic function is given by (1.1) with $m = 0$. The Lévy density of X is given by $J(x) = j(|x|) = \mathcal{A}(d, -\alpha)|x|^{-(d+\alpha)}$, where

$$\mathcal{A}(d, -\alpha) = \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\pi^{d/2}\Gamma(1 - \frac{\alpha}{2})}.$$

Here Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t}\, dt$ for every $\lambda > 0$.

The Lévy measure of X^m has a density

$$J^m(x) = j^m\big(|x|\big) = \mathcal{A}(d, -\alpha)|x|^{-d-\alpha}\psi\big(m^{1/\alpha}|x|\big) = j(|x|)\psi\big(m^{1/\alpha}|x|\big) \tag{2.1}$$

where

$$\psi(r) := 2^{-(d+\alpha)}\Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}}\, ds, \tag{2.2}$$

which is decreasing and a smooth function of r^2 satisfying $\psi(0) = 1$ and

$$\psi(r) \asymp \phi(r) := e^{-r}\big(1 + r^{(d+\alpha-1)/2}\big) \quad \text{on } [0, \infty) \tag{2.3}$$

(see [25, Lemma 2] and [14, pp. 276–277] for the details).

Put $J^m(x, y) := j^m(|x - y|)$. The Lévy density gives rise to a Lévy system for X^m, which describes the jumps of the process X^m: for any $x \in \mathbb{R}^d$, stopping time T (with respect to the filtration of X^m) and non-negative Borel function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$,

$$\mathbb{E}_x\left[\sum_{s\leqslant T} f\big(s, X^m_{s-}, X^m_s\big)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f\big(s, X^m_s, y\big) J^m\big(X^m_s, y\big)\, dy\right) ds\right]. \tag{2.4}$$

(See, for example, [13, Appendix A].)

We will use $p^m(t, x, y) = p^m(t, x - y)$ to denote the transition density of X^m. From (1.1), one can easily see that X^m has the following approximate scaling property: for every $b > 0$

$$\big\{b^{-1/\alpha}\big(X^{m/b}_{bt} - X^{m/b}_0\big),\ t \geqslant 0\big\} \text{ has the same distribution as that of } \big\{X^m_t - X^m_0,\ t \geqslant 0\big\}. \tag{2.5}$$

In terms of transition densities, this scaling property can be written as

$$p^m(t,x,y) = b^{d/\alpha} p^{m/b}\big(bt, b^{1/\alpha}x, b^{1/\alpha}y\big) \quad \text{for every } t, b > 0,\ x, y \in \mathbb{R}^d. \tag{2.6}$$

For any $m, c > 0$, we define a function $\widetilde{\Psi}_{d,\alpha,m,c}(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$\widetilde{\Psi}_{d,\alpha,m,c}(t,x,y) := \begin{cases} t^{-d/\alpha} \wedge t J^m(x,y), & \forall t \in (0, 1/m]; \\ m^{d/\alpha - d/2} t^{-d/2} \exp(-c^{-1}(m^{1/\alpha}|x-y| \wedge m^{2/\alpha - 1}\frac{|x-y|^2}{t})), & \forall t \in (1/m, \infty). \end{cases}$$

Using [7, Theorem 1.2], [10, Theorem 4.1] and (2.6) we get

Theorem 2.1. *There exist $c_1, C_1 > 1$ such that for all $m > 0$ and $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$c_1^{-1} \widetilde{\Psi}_{d,\alpha,m,1/C_1}(t,x,y) \leqslant p^m(t,x,y) \leqslant c_1 \widetilde{\Psi}_{d,\alpha,m,C_1}(t,x,y).$$

For any open set D, we use $\tau_D^m := \inf\{t > 0\colon\ X_t^m \notin D\}$ to denote the first exit time from D by X^m, and $X^{m,D}$ to denote the subprocess of X^m killed upon exiting D (or, the killed relativistic stable process in D with mass m). It is known (see [13]) that $X^{m,D}$ has a continuous transition density $p_D^m(t,x,y)$ with respect to the Lebesgue measure. $p_D^m(t,x,y)$ has the following scaling property:

$$p_D^m(t,x,y) = b^{d/\alpha} p^{m/b}_{b^{1/\alpha}D}\big(bt, b^{1/\alpha}x, b^{1/\alpha}y\big) \quad \text{for every } t, b > 0,\ x, y \in D. \tag{2.7}$$

Thus the Green function $G_D^m(x,y) := \int_0^\infty p_D^m(t,x,y)\,dt$ of $X^{m,D}$ satisfies

$$G_D^m(x,y) = b^{(d-\alpha)/\alpha} G^{m/b}_{b^{1/\alpha}D}\big(b^{1/\alpha}x, b^{1/\alpha}y\big) \quad \text{for every } b > 0,\ x, y \in D. \tag{2.8}$$

We now introduce the space–time process $Z_s^m := (V_s, X_s^m)$, where $V_s = V_0 - s$. The law of the space–time process $s \mapsto Z_s^m$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$ and as usual, $\mathbb{E}^{(t,x)}[\xi] = \int \xi(\omega)\,\mathbb{P}^{(t,x)}(d\omega)$.

We say that a non-negative Borel function $h(t,x)$ on $[0,\infty) \times \mathbb{R}^d$ is *parabolic* with respect to the process X^m in a relatively open subset E of $[0,\infty) \times \mathbb{R}^d$ if for every relatively compact open subset E_1 of E, $h(t,x) = \mathbb{E}^{(t,x)}[h(Z^m_{\widetilde{\tau}^m_{E_1}})]$ for every $(t,x) \in E_1$, where $\widetilde{\tau}^m_{E_1} = \inf\{s > 0\colon\ Z_s^m \notin E_1\}$. Note that $p_D^m(\cdot,\cdot,y)$ is parabolic with respect to the process X^m in $(0,\infty) \times D$.

The following uniform parabolic Harnack inequality is an extension of [10, Theorem 2.9] in that it is stated for all $r > 0$ and $m > 0$ instead of only for $r \in (0,R]$ and $m \in (0,M]$. Due to the recent result in [7], the following uniform parabolic Harnack inequality is an easy consequence of the approximate scaling property (2.5) and the parabolic Harnack inequality [7, Theorem 4.11].

Theorem 2.2. *For $M > 0$ and $\delta \in (0,1)$, there exists $c = c(d,\alpha,\delta,M) > 0$ such that for every $m > 0$, $x_0 \in \mathbb{R}^d$, $t_0 \geqslant 0$, $r > 0$ and every non-negative function u on $[0,\infty) \times \mathbb{R}^d$ that is parabolic with respect to the process X^m on $(t_0, t_0 + 4\delta(r^\alpha \vee m^{2/\alpha-1}r^2)] \times B(x_0, 4r)$,*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1) \leqslant c \inf_{(t_2,y_2)\in Q_+} u(t_2,y_2),$$

where $Q_- = [t_0+\delta(r^\alpha \vee m^{2/\alpha-1}r^2), t_0+2\delta(r^\alpha \vee m^{2/\alpha-1}r^2)] \times B(x_0,r)$ *and* $Q_+ = [t_0+3\delta(r^\alpha \vee m^{2/\alpha-1}r^2), t_0+4\delta(r^\alpha \vee m^{2/\alpha-1}r^2)] \times B(x_0,r)$.

We now prove the upper bound estimate in Theorem 1.2.

Proof of the upper bound estimate in Theorem 1.2. Without loss of generality, we assume $M = 1/3$ and $T = 1$. In view of Theorem 1.1(i), we only need to prove the upper bound in Theorem 1.2 for $t \geqslant 3$. By the semigroup property and Theorem 1.1(i), we have for $t \geqslant 3$, $0 < m \leqslant 1/3$ and $x, y \in D$,

$$\begin{aligned} p_D^m(t,x,y) &= \int_D \int_D p_D^m(1,x,z)p_D^m(t-2,z,w)p_D^m(1,w,y)\,dz\,dw \\ &\leqslant c_1\big(1\wedge\delta_D(x)\big)^{\alpha/2}\big(1\wedge\delta_D(y)\big)^{\alpha/2} f(t,x,y), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} f(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} &\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/(16))}{|x-z|^{d+\alpha}}\right)p^m(t-2,z,w) \\ &\cdot\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/(16))}{|w-y|^{d+\alpha}}\right)dz\,dw. \end{aligned}$$

By Theorem 2.1 and (2.3), there exists a constant $A \geqslant 16$ such that for every $t \geqslant 3$

$$\begin{aligned} f(t,x,y) &\leqslant c_2 \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(1\wedge\frac{\phi(A^{-1}m^{1/\alpha}|x-z|)}{|x-z|^{d+\alpha}}\right)p^m\big(t-2,A^{-1}z,A^{-1}w\big) \\ &\quad\cdot\left(1\wedge\frac{\phi(A^{-1}m^{1/\alpha}|w-y|)}{|w-y|^{d+\alpha}}\right)dz\,dw \\ &\leqslant c_3 \int_{\mathbb{R}^d\times\mathbb{R}^d} p^m\big(1,A^{-1}x,A^{-1}z\big)p^m\big(t-2,A^{-1}z,A^{-1}w\big)p^m\big(1,A^{-1}w,A^{-1}y\big)\,dz\,dw. \end{aligned}$$

Thus by the change of variables $\widehat{z} = A^{-1}z$, $\widehat{w} = A^{-1}w$ and the semigroup property, we have that

$$\begin{aligned} f(t,x,y) &\leqslant c_4 \int_{\mathbb{R}^d\times\mathbb{R}^d} p^m\big(1,A^{-1}x,\widehat{z}\big)p^m(t-2,\widehat{z},\widehat{w})p^m\big(1,\widehat{w},A^{-1}y\big)\,d\widehat{z}\,d\widehat{w} \\ &= c_4 p^m\big(t,A^{-1}x,A^{-1}y\big). \end{aligned} \tag{2.10}$$

Now using (2.7) and Theorem 2.1 again, we conclude that for every $m \leqslant 1/3$

$$\begin{aligned}
&p^m\big(t, A^{-1}x, A^{-1}y\big)\\
&\quad = A^d p^{mA^{-\alpha}}\big(A^\alpha t, x, y\big)\\
&\quad \leqslant c_5 \begin{cases} t^{-d/\alpha} \wedge t j^m(c_6|x-y|), & \forall t \in [3, 1/m];\\ m^{d/\alpha - d/2} t^{-d/2} \exp(-c_6(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t})), & \forall t > 1/m. \end{cases}
\end{aligned}$$

This together with (2.9) and (2.10) establishes the upper bound estimate in Theorem 1.2. □

3. Interior lower bound estimates

Throughout this section, we assume the dimension $d \geqslant 1$. We discuss interior lower bound estimates for the heat kernel $p_D^m(t, x, y)$ all $t > 0$. We first establish interior lower bound estimates for the heat kernel $p_D^m(t, x, y)$ of an arbitrary open set for all $m > 0$ and $t \leqslant T/m$.

Proposition 3.1. *Suppose that D is an arbitrary open set in $\mathbb{R}^d$ and $T > 0$ is a constant. There exists a constant $c = c(d, \alpha, T) > 0$ such that for all $m > 0$, $(t, x, y) \in (0, T/m] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant t^{1/\alpha}$, we have $p_D^m(t, x, y) \geqslant c(t^{-d/\alpha} \wedge t J^m(x, y))$.*

Proof. By [10, Proposition 3.5], there is a constant $c = c(d, \alpha, T) > 0$ such that for $m > 0$ and $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant t^{1/\alpha}$, we have $p_D^1(t, x, y) \geqslant c(t^{-d/\alpha} \wedge t J^1(x, y))$. The conclusion of the proposition for general $m > 0$ follows immediately from this and the scaling property (2.7). □

For notational convenience, we denote the ball $B(0, r)$ by B_r. In the rest of this section, we will establish interior lower bound estimate on the heat kernel $p^m_{\overline{B}_R^c}(t, x, y)$ for $m > 0$, $R > 0$, $t \geqslant T/m$, where T is a positive constant. To achieve this, we first establish some results for a large class of open sets which might be of independent interest.

Lemma 3.2. *For any positive constants T and a, there exists $c = c(d, \alpha, a, T) > 0$ such that for any $t \geqslant T$,*

$$\inf_{y\in\mathbb{R}^d} \mathbb{P}_y\big(\tau^1_{B(y, a\sqrt{t})} > t\big) \geqslant c.$$

Proof. This result is an easy consequence of [7, Theorem 4.8]. In fact, by [7, Theorem 4.8]

$$\begin{aligned}
\mathbb{P}_y\big(\tau^1_{B(y,a\sqrt{t})} > t\big) &= \int_{B(y,a\sqrt{t})} p^1_{B(y,a\sqrt{t})}(t, y, w)\, dw\\
&\geqslant \int_{B(y,a\sqrt{t}/2)} p^1_{B(y,a\sqrt{t})}(t, y, w)\, dw \geqslant c. \qquad \square
\end{aligned}$$

Lemma 3.3. *Let a and T be positive constants. There exist $c_i = c_i(d, \alpha, T, a) > 0$, $i = 1, 2$, such that for all $(t, x, y) \in [T, \infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$ and $|x - y| \geqslant 2^{-1}\sqrt{t}$,*

$$\mathbb{P}_x\big(X_t^{1,D} \in B\big(y, (a \wedge 1)2^{-1}\sqrt{t}\big)\big) \geqslant c_1 \frac{t^{1+d/2} e^{-c_2|x-y|}}{|x-y|^{d+\alpha}}.$$

Proof. By Lemma 3.2, starting at $z \in B(y, (a \wedge 1)\sqrt{t}/4)$, with probability at least $c_1 = c_1(d, \alpha, T, a) > 0$, the process X^1 does not move more than $(a \wedge 1)6^{-1}\sqrt{t}$ by time t. Thus, it is sufficient to show that there exist constants $c_2 = c_2(d, \alpha, T, a) > 0$, $i = 2, 3$, such that for any $t \geqslant T$ and $(x, y) \in D \times D$ with $|x - y| \geqslant \sqrt{t}/2$,

$$\mathbb{P}_x\big(X^{1,D} \text{ hits the ball } B\big(y, (a \wedge 1)\sqrt{t}/4\big) \text{ by time } t\big) \geqslant c_2 \frac{t^{1+d/2} e^{-c_3|x-y|}}{|x-y|^{d+\alpha}}.$$

Let $B_x := B(x, (a \wedge 1)6^{-1}\sqrt{t})$, $B_y := B(y, (a \wedge 1)6^{-1}\sqrt{t})$ and $\tau_x^1 := \tau_{B_x}^1$. It follows from Lemma 3.2 that there exists $c_4 = c_4(d, \alpha, a, T) > 0$ such that

$$\mathbb{E}_x\big[t \wedge \tau_x^1\big] \geqslant t\mathbb{P}_x\big(\tau_x^1 \geqslant t\big) \geqslant c_4 t \quad \text{for } t \geqslant T. \tag{3.1}$$

By the Lévy system in (2.4),

$$\begin{aligned}
&\mathbb{P}_x\big(X^{1,D} \text{ hits the ball } B\big(y, (a \wedge 1)\sqrt{t}/4\big) \text{ by time } t\big) \\
&\quad \geqslant \mathbb{P}_x\big(X^1_{t\wedge\tau_x^1} \in B\big(y, (a \wedge 1)\sqrt{t}/4\big) \text{ and } t \wedge \tau_x^1 \text{ is a jumping time}\big) \\
&\quad \geqslant \mathbb{E}_x\Bigg[\int_0^{t\wedge\tau_x^1} \int_{B_y} J^1\big(X_s^1, u\big)\, du\, ds\Bigg].
\end{aligned}$$

Since for any $(z, u) \in B_x \times B_y$ we have

$$|z - u| \leqslant |z - x| + |x - y| + |y - u| \leqslant (a \wedge 1)\sqrt{t}/3 + |x - y| \leqslant \big(1 + 2(a \wedge 1)/3\big)|x - y|,$$

we get that

$$\int_{B_y} J^1\big(X_s^1, u\big)\, du \geqslant c_5 |B_y| \frac{e^{-c_6|x-y|}}{|x-y|^{d+\alpha}} \quad \text{for every } s < t \wedge \tau_x^1.$$

Thus by (3.1),

$$\begin{aligned}
\mathbb{P}_x\big(X^{1,D} \text{ hits the ball } B\big(y, (a \wedge 1)\sqrt{t}/4\big) \text{ by time } t\big) &\geqslant c_7 \mathbb{E}_x[t \wedge \tau_x]|B_y| \frac{e^{-c_6|x-y|}}{|x-y|^{d+\alpha}} \\
&\geqslant c_8 \frac{t^{1+d/2} e^{-c_6|x-y|}}{|x-y|^{d+\alpha}}. \qquad \square
\end{aligned}$$

For an open set $D \subset \mathbb{R}^d$ and $(\lambda_1, \lambda_2) \in (1, \infty) \times (0, \infty)$, we say *the path distance in D is comparable to the Euclidean distance with characteristics* (λ_1, λ_2) if the following holds for any $r > 0$: for every x, y in the same component of D with $\delta_D(x) \wedge \delta_D(y) \geqslant r$, there is a length parameterized rectifiable curve l in D connecting x to y so that the length of l is no larger than $\lambda_1|x - y|$ and $\delta_D(l(u)) \geqslant \lambda_2 r$, $u \in [0, |l|]$.

Clearly, such a property holds for all Lipschitz domains with compact complements and domains above graphs of Lipschitz functions.

Proposition 3.4. *Suppose that D is a domain such that the path distance in D is comparable to the Euclidean distance with characteristics (λ_1, λ_2). For any positive constants a and T, there exists a positive constant $c = c(d, \alpha, T, a, \lambda_1, \lambda_2)$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$ and $\sqrt{t} \geqslant 2|x-y|$, we have $p_D^1(t, x, y) \geqslant ct^{-d/2}$.*

Proof. Let $t \geqslant T$ and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$ and $\sqrt{t} \geqslant 2|x-y|$. The assumption that D is a domain such that the path distance in D is comparable to the Euclidean distance with characteristics (λ_1, λ_2) enables us to apply the parabolic Harnack inequality (Theorem 2.2) $N = N(a, \lambda_1, \lambda_2)$ times and to get that there exists $c_1 = c_1(d, \alpha, T, a, \lambda_1, \lambda_2) > 0$ such that

$$p_D^1(t/2, x, w) \leqslant c_1 p_D^1(t, x, y) \quad \text{for } w \in B\big(x, 2(a \wedge 1)\sqrt{t}/3\big).$$

This together with Lemma 3.2 yields that

$$\begin{aligned} p_D^1(t, x, y) &\geqslant \frac{1}{c_1 |B(x, (a \wedge 1)\sqrt{t}/2)|} \int_{B(x, (a\wedge 1)\sqrt{t}/2)} p_D^1(t/2, x, w)\, dw \\ &\geqslant c_2 t^{-d/2} \mathbb{P}_x\big(\tau^1_{B(x,(a\wedge 1)\sqrt{t}/2)} > t/2\big) \geqslant c_3 t^{-d/2} \end{aligned}$$

where $c_i = c_i(d, \alpha, T, a, \lambda_1, \lambda_2) > 0$, $i = 2, 3$. □

Proposition 3.5. *Suppose that D is a domain such that the path distance in D is comparable to the Euclidean distance with characteristics (λ_1, λ_2). For any positive constants a and T, there exist constants $c_i = c_i(d, \alpha, a, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$, such that*

$$p_D^1(t, x, y) \geqslant c_1 \frac{t e^{-c_2|x-y|}}{|x-y|^{d+\alpha}} \tag{3.2}$$

for every $(t, x, y) \in [T, \infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$ and $|x-y|^2 \geqslant t/8$.

Proof. By the semigroup property, Proposition 3.4 and Lemma 3.3, there exist positive constants $c_i = c_i(d, \alpha, T, a)$, $i = 1, 2, 3$, such that

$$\begin{aligned} p_D^1(t, x, y) &\geqslant \int_{B(y, (a\wedge 1)2^{-1}(t/2)^{1/2})} p_D^1(t/2, x, z) p_D^1(t/2, z, y)\, dz \\ &\geqslant c_1 t^{-d/2} \mathbb{P}_x\big(X^{1,D}_{t/2} \in B\big(y, (a \wedge 1)2^{-1}(t/2)^{1/2}\big)\big) \geqslant c_2 \frac{t e^{-c_3|x-y|}}{|x-y|^{d+\alpha}}. \end{aligned}$$ □

Theorem 3.6. *Suppose that D is a domain such that the path distance in D is comparable to the Euclidean distance with characteristics (λ_1, λ_2). For any $C^*, a > 0$, there exist $c_i = c_i(d, \alpha, a, C^*, \lambda_1, \lambda_2) > 0$, $i = 1, 2$, such that for every $t \in (0, \infty)$ and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$,*

$$p_D^1(t, x, y) \geqslant c_1 t^{-d/2} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \quad \text{when } C^*|x-y| \leqslant t \leqslant |x-y|^2.$$

Proof. Fix $C^* > 0$. Suppose that x, y are in D with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$ and satisfy $C^*|x-y| \leqslant t \leqslant |x-y|^2$. For simplicity, let $R := |x-y|$. Note that $t \geqslant (C^*)^2$.

By our assumption on D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 R$ and $\delta_D(l(u)) \geqslant \lambda_2 a\sqrt{t}$ for every $u \in [0, |l|]$. Let $\lambda_3 \geqslant \max\{4/(\lambda_2^2 a^2), (12\lambda_1)^2\}$ and k the smallest integer satisfying $k \geqslant \lambda_3 R^2/t$. (The integer k depends on t and R.) Then, since $t \in [C^* R, R^2]$,

$$\frac{t}{k} \geqslant \frac{t}{1+\lambda_3 R^2/t} = \frac{t^2}{t+\lambda_3 R^2} \geqslant \frac{(t/R)^2}{1+\lambda_3} \geqslant \frac{(C^*)^2}{1+\lambda_3}. \tag{3.3}$$

Let $x_j = l(j|l|/k)$ and $B_j := B(x_j, \sqrt{t/k}/8)$, $j = 0, 1, \ldots, k$. Note that, since $\lambda_2^2 a^2/4 \leqslant \lambda_3 \leqslant \lambda_3 R^2/t \leqslant k$, we have $\delta_D(x_j) > \lambda_2 a\sqrt{t} \geqslant 2\sqrt{t/k}$ for each j. So we have $B_j \subset D$ and for each $y \in B_j$, $B(y, \sqrt{t/k}) \subset D$.

Observe that for $(y_j, y_{j+1}) \in B_j \times B_{j+1}$, since $\lambda_3 > (12\lambda_1)^2$,

$$\begin{aligned} |y_j - y_{j+1}| &\leqslant |x_j - x_{j+1}| + |y_j - x_j| + |y_{j+1} - x_{j+1}| \leqslant \frac{|l|}{k} + \frac{1}{4}\sqrt{t/k} \leqslant \frac{\lambda_1}{\sqrt{k}}\frac{R}{\sqrt{k}} + \frac{1}{4}\sqrt{t/k} \\ &\leqslant \frac{\lambda_1}{\sqrt{k}}\frac{R\sqrt{t}}{\sqrt{\lambda_3}R} + \frac{1}{4}\sqrt{t/k} = (\lambda_1/\sqrt{\lambda_3} + 1/4)\sqrt{t/k} < \sqrt{t/k}/3. \end{aligned} \tag{3.4}$$

Now using (3.3), (3.4) and Proposition 3.4, we get

$$p_D^1(t/k, y_j, y_{j+1}) \geqslant c_1 (t/k)^{-d/2}, \quad \text{for every } (y_j, y_{j+1}) \in B_j \times B_{j+1}. \tag{3.5}$$

Using (3.5) and the fact $k \geqslant \lambda_3 R^2/t$, we have

$$\begin{aligned} p_D^1(t,x,y) &\geqslant \int_{B_1} \cdots \int_{B_{l-1}} p_D^1(t/k, x, y_1) \cdots p_D^1(t/k, y_{k-1}, y)\, dy_1 \ldots dy_{k-1} \\ &\geqslant c_1 (t/k)^{-d/2} \prod_{i=1}^{k-1} \bigl(c_1 8^{-d} \bigl|B(0,1)\bigr| (t/k)^{-d/2} (t/k)^{d/2}\bigr) \\ &= c_1 (t/k)^{-d/2} \bigl(c_1 8^{-d} \bigl|B(0,1)\bigr|\bigr)^{k-1} \\ &\geqslant c_2 (t/k)^{-d/2} \exp(-c_3 k) \geqslant c_4 t^{-d/2} \exp\biggl(-\frac{c_5|x-y|^2}{t}\biggr). \qquad \square \end{aligned}$$

Combining Theorem 3.6 with Propositions 3.4 and 3.5, we have the following lower bound estimates for $p_D^1(t,x,y)$.

Theorem 3.7. *Let a and T be positive constants. Suppose that D is a domain such that the path distance in D is comparable to the Euclidean distance with characteristics (λ_1, λ_2). Then there exist constants $c_i = c_i(d, \alpha, a, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$, such that for every $(t, x, y) \in [T, \infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geqslant a\sqrt{t}$,*

$$p_D^1(t,x,y) \geqslant c_1 t^{-d/2} \exp\biggl(-c_2\biggl(|x-y| \wedge \frac{|x-y|^2}{t}\biggr)\biggr).$$

Observe that any exterior ball $\overline{B}^c$ is a domain in which the path distance is comparable to the Euclidean distance with characteristics (λ_1, λ_2) independent of the radius of the ball B. The following follows immediately from Theorem 3.7 and the scaling property (2.7).

Theorem 3.8. *Let a and T be positive constants. Then there exist constants $c_i = c_i(d, \alpha, a, T) > 0$, $i = 1, 2$, such that for every $R > 0$, $m > 0$ and $(t, x, y) \in [T/m, \infty) \times \overline{B}^c_R \times \overline{B}^c_R$ with $\delta_{\overline{B}^c_R}(x) \wedge \delta_{\overline{B}^c_R}(y) \geqslant a m^{1/2-1/\alpha}\sqrt{t}$,*

$$p^m_{\overline{B}^c_R}(t, x, y) \geqslant c_1 m^{d/\alpha-d/2} t^{-d/2} \exp\left(-c_2\left(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\right)\right).$$

4. Small time lower bound estimates

In the remainder of this paper we will always assume that the dimension $d \geqslant 3$. The goal of this section is to establish the lower bound estimates in Theorem 1.2 for $t \leqslant T/m$, where T is a positive constant.

Let $G^m(x, y)$ be the Green function of X^m. It follows from [24, Theorems 3.1 and 3.3] that there exists $c = c(d, \alpha) > 1$ such that

$$c^{-1}\left(|x-y|^{\alpha-d} + |x-y|^{2-d}\right) \leqslant G^1(x, y) \leqslant c\left(|x-y|^{\alpha-d} + |x-y|^{2-d}\right).$$

Using this and (2.8) we get that for every $m > 0$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &c^{-1}\left(|x-y|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x-y|^{2-d}\right) \\ &\quad \leqslant G^m(x, y) \leqslant c\left(|x-y|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x-y|^{2-d}\right). \end{aligned} \tag{4.1}$$

For a Borel set A, we use σ^m_A to denote the first hitting time of A by X^m. Recall that we denote the ball $B(0, r)$ by B_r.

Lemma 4.1. *There is a constant $C_2 = C_2(d, \alpha) > 1$ such that for all $R, m > 0$,*

$$\begin{aligned} &C_2^{-1}\frac{R^d}{R^\alpha + m^{(2-\alpha)/\alpha}R^2}\left(|x|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x|^{2-d}\right) \\ &\quad \leqslant \mathbb{P}_x\left(\sigma^m_{\overline{B}_R} < \infty\right) \leqslant C_2\frac{R^d}{R^\alpha + m^{(2-\alpha)/\alpha}R^2}\left(|x|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x|^{2-d}\right) \quad \text{for } |x| \geqslant 2R. \end{aligned}$$

Proof. For $|x| \geqslant 2R$,

$$\begin{aligned} \int_{\overline{B}_R} G^m(x, y)\, dy &\asymp \int_{\overline{B}_R} \left(|x-y|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x-y|^{2-d}\right) dy \\ &\asymp R^d\left(|x|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x|^{2-d}\right). \end{aligned} \tag{4.2}$$

On the other hand, for $|z| \leqslant R$,

$$\int_{\overline{B}_R} G^m(z,y)\,dy \asymp \int_{\overline{B}_R} \big(|z-y|^{\alpha-d} + m^{(2-\alpha)/\alpha}|z-y|^{2-d}\big)\,dy \asymp R^\alpha + m^{(2-\alpha)/\alpha} R^2.$$

Thus, by the strong Markov property of X^m, for $|x| > 2R$,

$$\int_{\overline{B}_R} G^m(x,y)\,dy = \mathbb{E}_x\bigg[\int_{\overline{B}_R} G^m\big(X^m_{\sigma^m_{\overline{B}_R}}, y\big)\,dy; \sigma^m_{\overline{B}_R} < \infty\bigg]$$

$$\asymp \big(R^\alpha + m^{(2-\alpha)/\alpha} R^2\big)\mathbb{P}_x\big(\sigma^m_{\overline{B}_R} < \infty\big). \tag{4.3}$$

Combining (4.2) and (4.3), we arrive at the conclusion of the lemma. □

The above lemma quantifies the transience of X^m when dimension $d \geqslant 3$, which in particular implies that for a compact set K and a point x far away from the origin, with large probability the process started at x will never visit K.

Lemma 4.2. *Suppose that a and T are positive constants. There exist constants $\varepsilon = \varepsilon(d,\alpha,a,T) > 0$ and $L_1 = L_1(d,\alpha,a,T) \geqslant 10^{4/\alpha}$ such that the following holds: for all $R > 0$, $m > 0$, $t \in (0, T/m]$, $|x| > L_1 R$ and $y \in B(x, at^{1/\alpha}) \cap \overline{B}^c_R$,*

$$\mathbb{P}_x\big(X^{m,\overline{B}^c_R}_t \in B\big(y, (t/2)^{1/\alpha}\big)\big) \geqslant \varepsilon.$$

Proof. Suppose that $t \in (0, T/m]$ and $y \in B(x, at^{1/\alpha}) \cap \overline{B}^c_R$. It follows from Theorem 2.1 that there exists $c_1 = c_1(T) > 0$ such that

$$\mathbb{P}_x\big(X^m_t \in B\big(y, (t/2)^{1/\alpha}\big)\big) \geqslant \inf_{w \in B(0, at^{1/\alpha})} \mathbb{P}_w\big(X^m_t \in B\big(0, (t/2)^{1/\alpha}\big)\big)$$

$$\geqslant c_1 \inf_{w \in B(0, at^{1/\alpha})} \int_{B(0,(t/2)^{1/\alpha})} \big(t^{-d/\alpha} \wedge t J^m(z,w)\big)\,dz.$$

Since for $w \in B(0, at^{1/\alpha})$ and $z \in B(0, t^{1/\alpha})$, $m^{1/\alpha}|z-w| \leqslant m^{1/\alpha}((a+1)t^{1/\alpha}) \leqslant (a+1)T^{1/\alpha}$, we have in view of (2.1) that

$$\mathbb{P}_x\big(X^m_t \in B\big(y, (t/2)^{1/\alpha}\big)\big) \geqslant c_1 \inf_{w \in B(0, at^{1/\alpha})} \int_{B(0,(t/2)^{1/\alpha})} \bigg(t^{-d/\alpha} \wedge \frac{t\psi((a+1)T^{1/\alpha})}{|z-w|^{d+\alpha}}\bigg)\,dz$$

$$\geqslant c_2 \inf_{w \in B(0,a)} \int_{B(0,(1/2)^{1/\alpha})} \bigg(1 \wedge \frac{1}{|w-z|^{d+\alpha}}\bigg)\,dz$$

$$\geqslant c_3 (a+1)^{-d-\alpha} \big|B\big(0, (1/2)^{1/\alpha}\big)\big| =: 2\varepsilon.$$

Thus for $x \in \mathbb{R}^d$ and $y \in B(x, at^{1/\alpha})$, we have

$$\mathbb{P}_x\big(X_t^m \notin B\big(y, (t/2)^{1/\alpha}\big)\big) = 1 - \mathbb{P}_x\big(X_t^m \in B\big(y, (t/2)^{1/\alpha}\big)\big) \leqslant 1 - 2\varepsilon. \tag{4.4}$$

Since $d \geqslant 3$, we may choose $L_1 \geqslant 10^{4/\alpha}$ so that $C_2(L_1^{2-d} + L_1^{\alpha-d}) \leqslant \varepsilon$. By Lemma 4.1, for all x with $|x| > L_1 R$ we have

$$\begin{aligned}
\mathbb{P}_x\big(\tau^m_{\overline{B}^c_R} \leqslant t\big) \leqslant \mathbb{P}_x\big(\sigma^m_{\overline{B}_R} < \infty\big) &\leqslant C_2 \frac{R^d}{R^\alpha + m^{(2-\alpha)/\alpha} R^2}\big(|x|^{\alpha-d} + m^{(2-\alpha)/\alpha}|x|^{2-d}\big) \\
&\leqslant C_2\left(\frac{R^\alpha}{R^\alpha + m^{(2-\alpha)/\alpha} R^2} L_1^{\alpha-d} + \frac{R^2}{m^{-(2-\alpha)/\alpha} R^\alpha + R^2} L_1^{2-d}\right) \\
&\leqslant C_2\big(L_1^{\alpha-d} + L_1^{2-d}\big) \leqslant \varepsilon.
\end{aligned} \tag{4.5}$$

Hence, combining (4.4) and (4.5) gives

$$\begin{aligned}
\mathbb{P}_x\big(X_t^{m,\overline{B}^c_R} \in B\big(y, (t/2)^{1/\alpha}\big)\big) &\geqslant \mathbb{P}_x\big(\tau^m_{\overline{B}^c_R} > t\big) - \mathbb{P}_x\big(X_t^{m,\overline{B}^c_R} \notin B\big(y, (t/2)^{1/\alpha}\big); \tau^m_{\overline{B}^c_R} > t\big) \\
&\geqslant \mathbb{P}_x\big(\tau^m_{\overline{B}^c_R} > t\big) - \mathbb{P}_x\big(X_t^m \notin B\big(y, (t/2)^{1/\alpha}\big)\big) \\
&\geqslant (1-\varepsilon) - (1-2\varepsilon) = \varepsilon. \qquad \Box
\end{aligned}$$

Lemma 4.3. *Let $T > 0$ be a constant and $L_1 = L_1(d, \alpha, 3, T)$ be the constant in Lemma 4.2. There exists constant $c = c(T, d, \alpha) > 0$ such that for all $m > 0$, $R > 0$, $t \in (0, T/m]$, x, y satisfying $|x| > L_1 R$, $|y| > L_1 R$ with $|x - y| \leqslant (t/6)^{1/\alpha}$, we have $p^m_{\overline{B}^c_R}(t, x, y) \geqslant ct^{-d/\alpha}$.*

Proof. Assume without loss of generality that $|y| \geqslant |x|$. If $\delta_{\overline{B}^c_R}(y) \geqslant (2t)^{1/\alpha}$, then

$$\delta_{\overline{B}^c_R}(x) \geqslant \delta_{\overline{B}^c_R}(y) - |x - y| \geqslant \big(2^{1/\alpha} - 6^{-1/\alpha}\big)t^{1/\alpha} > t^{1/\alpha},$$

and hence the lemma follows immediately from Proposition 3.1.

Now assume $\delta_{\overline{B}^c_R}(y) < (2t)^{1/\alpha}$. Since $t - 4^{-1}\delta_{\overline{B}^c_R}(y)^\alpha > 4^{-1}\delta_{\overline{B}^c_R}(y)^\alpha$, by the semigroup property and Theorem 2.2 we have

$$\begin{aligned}
p^m_{\overline{B}^c_R}(t, x, y) &\geqslant \int\limits_{B(y, 8^{-1/\alpha}\delta_{\overline{B}^c_R}(y))} p^m_{\overline{B}^c_R}\big(4^{-1}\delta_{\overline{B}^c_R}(y)^\alpha, x, z\big) p^m_{\overline{B}^c_R}\big(t - 4^{-1}\delta_{\overline{B}^c_R}(y)^\alpha, z, y\big)\, dz \\
&\geqslant c_1 \mathbb{P}_x\big(X^{m,\overline{B}^c_R}_{4^{-1}\delta_{\overline{B}^c_R}(y)^\alpha} \in B\big(y, 8^{-1/\alpha}\delta_{\overline{B}^c_R}(y)\big)\big) p^m_{\overline{B}^c_R}\left(t - \frac{3}{8}\delta_{\overline{B}^c_R}(y)^\alpha, y, y\right).
\end{aligned} \tag{4.6}$$

Observe that $|x - y| \leqslant 2|y| = 2(\delta_{\overline{B}^c_R}(y) + R) < 3\delta_{\overline{B}^c_R}(y)$, where the last inequality follows because $|y| > L_1 R \geqslant 3R$ implies $\delta_{\overline{B}^c_R}(y) > 2R$. Thus, $y \in B(x, 3\delta_{\overline{B}^c_R}(y)) \cap \overline{B}^c_R$ and Lemma 4.2 gives

$$\mathbb{P}_x\big(X^{m,\overline{B}^c_R}_{4^{-1}\delta_{\overline{B}^c_R}(y)^\alpha} \in B\big(y, 4^{-1/\alpha}\delta_{\overline{B}^c_R}(y)\big)\big) \geqslant \varepsilon.$$

To bound the second term in (4.6), we let $s := t - 3 \cdot 8^{-1}\delta_{\overline{B}^c_R}(y)^\alpha$ and note that $s < t \leqslant T/m$. Thus, by the semigroup property, the Cauchy–Schwarz inequality and Lemma 4.2,

$$\begin{aligned} p^m_{\overline{B}^c_R}(s, y, y) &\geqslant \int\limits_{B(y,(s/4)^{1/\alpha})} \big(p^m_{\overline{B}^c_R}(s/2, y, z)\big)^2\,dz \\ &\geqslant \frac{1}{|B(y, (s/4)^{1/\alpha})|}\big(\mathbb{P}_y\big(X^{m,D}_{s/2} \in B\big(y, (s/4)^{1/\alpha}\big)\big)\big)^2 \geqslant c_2 s^{-d/\alpha} \geqslant c_2 t^{-d/\alpha}. \end{aligned}$$

The proof is now complete. $\square$

Proposition 4.4. *Let $T > 0$ be a constant and $L_1 = L_1(d, \alpha, 3, T)$ be the constant in Lemma* 4.2. *There is a constant $c = c(d, \alpha, T) > 0$ such that for every $m > 0$, $R > 0$, $t \in (0, T/m]$ and x, y with $|x| > L_1 R$ and $|y| > L_1 R$,*

$$p^m_{\overline{B}^c_R}(t, x, y) \geqslant c\big(t^{-d/\alpha} \wedge t j^m\big(2|x - y|\big)\big).$$

Proof. Let $|x| > L_1 R$, $|y| > L_1 R$ and $t \in (0, T/m]$. By Lemma 4.3, we only need to show that

$$p^m_{\overline{B}^c_R}(t, x, y) \geqslant c_1\big(t^{-d/\alpha} \wedge t j^m\big(2|x - y|\big)\big) \quad \text{when } |x - y| > (t/6)^{1/\alpha}. \tag{4.7}$$

If $t < 60 \cdot 4^\alpha R^\alpha$,

$$\delta_{\overline{B}^c_R}(x) \wedge \delta_{\overline{B}^c_R}(y) \geqslant (L_1 - 1)R \geqslant \big(10^{4/\alpha} - 1\big)R \geqslant 60^{1/\alpha} \cdot 4R > t^{1/\alpha}.$$

Thus, by Proposition 3.1, (4.7) is true in this case.

Now we assume that $t \geqslant 60 \cdot 4^\alpha R^\alpha$. Since one of $|x|$ and $|y|$ should be no less than $|x - y|/2$, we assume without loss of generality that $|y| \geqslant |x - y|/2$. Let $x_0 := x + (t/60)^{1/\alpha} x/|x|$. Note that $B(x_0, (t/60)^{1/\alpha}) \subset \overline{B}^c_R$. Since $|x - y| > (t/6)^{1/\alpha}$, we get for every $z \in B(x_0, (t/60)^{1/\alpha}/4)$, $|x - z| \leqslant |x_0 - z| + (t/60)^{1/\alpha} < (t/12)^{1/\alpha}$ and $|z - y| \leqslant |x - y| + |x - z| < |x - y| + (t/12)^{1/\alpha} < 2|x - y|$. Moreover,

$$\delta_{\overline{B}^c_R}(y) = |y| - R \geqslant \frac{1}{2}|x - y| - \frac{1}{4}(t/60)^{1/\alpha} \geqslant \frac{1}{2}(t/6)^{1/\alpha} - \frac{1}{4}(t/60)^{1/\alpha} > \frac{1}{4}(t/6)^{1/\alpha}$$

while for $z \in B(x_0, \frac{1}{4}(t/60)^{1/\alpha})$, we have $|z| \geqslant |x| > L_1 R$ and

$$\begin{aligned} \delta_{\overline{B}^c_R}(z) &= |z| - R \geqslant |x_0| - |x_0 - z| - \frac{1}{4}(t/60)^{1/\alpha} \\ &\geqslant |x| + (t/60)^{1/\alpha} - \frac{1}{4}(t/60)^{1/\alpha} - \frac{1}{4}(t/60)^{1/\alpha} \geqslant \frac{1}{2}(t/60)^{1/\alpha}. \end{aligned}$$

Let $a = 4^{-\alpha}(60)^{-1}$. By the semigroup property, Proposition 3.1 and Lemma 4.3, there exist positive constants $c_i = c_i(d, \alpha, T)$ for $i = 2, 3$, such that

$$
\begin{aligned}
p^m_{\overline{B}^c_R}(t,x,y) &= \int_{\overline{B}^c_R} p^m_{\overline{B}^c_R}\big((1-a)t,x,z\big)p^m_{\overline{B}^c_R}(at,z,y)\,dz \\
&\geqslant \int_{B(x_0,\frac{1}{4}(t/60)^{1/\alpha})} p^m_{\overline{B}^c_R}\big((1-a)t,x,z\big)p^m_{\overline{B}^c_R}(at,z,y)\,dz \\
&\geqslant c_2 \int_{B(x_0,\frac{1}{4}(t/60)^{1/\alpha})} t^{-d/\alpha}\big(t^{-d/\alpha}\wedge tJ^m(z,y)\big)\,dz \\
&\geqslant c_3\big(t^{-d/\alpha}\wedge tj^m\big(2|x-y|\big)\big).
\end{aligned}
$$

This proves (4.7). □

We recall the following lemma from [11].

Lemma 4.5. *(See [11, Lemma 2.2].) Let λ, T, M be fixed positive constants. Suppose $x, x_0 \in \mathbb{R}^d$ satisfy $|x-x_0| = \lambda T^{1/\alpha}$. Then for all $a \in (0, M]$ and $z \in \mathbb{R}^d$,*

$$
T^{-d/\alpha} \wedge \frac{T\phi(a|x-z|)}{|x-z|^{d+\alpha}} \asymp T^{-d/\alpha} \wedge \frac{T\phi(a|x_0-z|)}{|x_0-z|^{d+\alpha}}, \tag{4.8}
$$

where the (implicit) comparison constants in (4.8) depend only on d, α, M, λ and T.

Theorem 4.6. *Let T, M and R be positive constants. Suppose that D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (r_0, Λ_0) and $D^c \subset B(0,R)$. Then there is a positive constant $c = c(d,\alpha,r_0,\Lambda_0,R,M,T)$ so that for all $0 < m \leqslant M$ and $(t,x,y) \in (0,T/m] \times D \times D$,*

$$
p^m_D(t,x,y) \geqslant c\left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\big(t^{-d/\alpha} \wedge tj^m\big(4|x-y|\big)\big).
$$

Proof. Without loss of generality, we assume $M = 1/3$ and $T = 1$. By Theorem 1.1, we only need to show the theorem for $t > 3$. For x and y in D, let $v \in \mathbb{R}^d$ be any unit vector satisfying $x \cdot v \geqslant 0$ and $y \cdot v \geqslant 0$. Let $L_1 = L_1(d,\alpha,3,1)$ be the constant given by Lemma 4.2 and define

$$
x_0 := x + L_1^2 Rv \quad \text{and} \quad y_0 := y + L_1^2 Rv. \tag{4.9}
$$

Then $|x_0|^2 = |x|^2 + (L_1^2R)^2 + 2L_1^2Rx \cdot v \geqslant (L_1^2R)^2$, and similarly, $|y_0|^2 \geqslant (L_1^2R)^2$.

Using the semigroup property and Theorem 1.1(i), for every $m \in (0, 1/3]$ and $t \in [3, 1/m]$ we have

$$
\begin{aligned}
p^m_D(t,x,y) &= \int_D\int_D p^m_D(1,x,z)p^m_D(t-2,z,w)p^m_D(1,w,y)\,dz\,dw \\
&\geqslant c_1\big(1 \wedge \delta_D(x)\big)^{\alpha/2}\big(1 \wedge \delta_D(y)\big)^{\alpha/2} f_1(t,x,y),
\end{aligned} \tag{4.10}
$$

where

$$f_1(t,x,y)=\int_{D\times D}\big(1\wedge\delta_D(z)\big)^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|)}{|x-z|^{d+\alpha}}\right)p_D^m(t-2,z,w)\big(1\wedge\delta_D(w)\big)^{\alpha/2}\cdot\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|)}{|w-y|^{d+\alpha}}\right)dz\,dw.$$

Note that by Proposition 4.4, for $z,w\in B(0,L_1R)^c$ and $t\in[3,1/m]$,

$$p_D^m(t-2,z,w)\geqslant p_{B_R^c}^m(t-2,z,w)\geqslant c_2\big((t-2)^{-d/\alpha}\wedge(t-2)j^m\big(2|z-w|\big)\big).$$

Since $m\leqslant 3$, the lower bound estimate in Theorem 1.1(i) and the above display together with Lemma 4.5 imply that

$$\begin{aligned}
f_1(t,x,y)&\geqslant c_3\int_{D\times D}\big(1\wedge\delta_D(z)\big)^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)p_D^m(t-2,z,w)\big(1\wedge\delta_D(w)\big)^{\alpha/2}\\
&\quad\cdot\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dz\,dw\\
&\geqslant c_4\int_{B(0,L_1R)^c\times B(0,L_1R)^c}\big(1\wedge\delta_D(z)\big)^{\alpha/2}\big(1\wedge\delta_D(w)\big)^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)\\
&\quad\cdot\big((t-2)^{-d/\alpha}\wedge(t-2)j^m\big(2|z-w|\big)\big)\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dz\,dw\\
&\geqslant c_5\int_{B(0,L_1R)^c\times B(0,L_1R)^c}\left(1\wedge\frac{\phi(2m^{1/\alpha}|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)\\
&\quad\cdot\big((t-2)^{-d/\alpha}\wedge(t-2)j^m\big(2|z-w|\big)\big)\left(1\wedge\frac{\phi(2m^{1/\alpha}|w-y_0|)}{|w-y_0|^{d+\alpha}}\right)dz\,dw.
\end{aligned}$$

Thus by the change of variables $\widehat{z}=2z$, $\widehat{w}=2w$, and Theorem 2.1, we have that

$$\begin{aligned}
f_1(t,x,y)&\geqslant c_6\int_{B(0,2L_1R)^c\times B(0,2L_1R)^c}\left(1\wedge\frac{\phi(m^{1/\alpha}|2x_0-\widehat{z}|)}{|2x_0-\widehat{z}|^{d+\alpha}}\right)p^m(t-2,\widehat{z},\widehat{w})\\
&\quad\cdot\left(1\wedge\frac{\phi(m^{1/\alpha}|\widehat{w}-2y_0|)}{|\widehat{w}-2y_0|^{d+\alpha}}\right)d\widehat{z}\,d\widehat{w}\\
&\geqslant c_7\int_{B(0,2L_1R)^c\times B(0,2L_1R)^c}p_{B(0,2L_1R)^c}^m(1,2x_0,\widehat{z})p_{B(0,2L_1R)^c}^m(t-2,\widehat{z},\widehat{w})\\
&\quad\cdot p_{B(0,2L_1R)^c}^m(1,\widehat{w},2y_0)\,dz\,dw\\
&=c_7p_{B(0,2L_1R)^c}^m(t,2x_0,2y_0).
\end{aligned}$$

We conclude from (2.7) and Proposition 4.4 that (recall that $|x_0|, |y_0| \geqslant L_1(L_1 R)$)

$$p^m_{B(0,2L_1R)^c}(t, 2x_0, 2y_0) = 2^{-d} p^{m2^\alpha}_{B(0,L_1R)^c}\big(2^{-\alpha}t, x_0, y_0\big) \geqslant c_8\big(t^{-d/\alpha} \wedge t j^1\big(4m^{1/\alpha}|x-y|\big)\big).$$

Combining the last two displays with (4.10) completes the proof. □

5. Large time lower bound estimates

The goal of this section is to establish the lower bound estimates in Theorem 1.2 for $t \geqslant T/m$, where T is a positive constant. For any $y \in \mathbb{R}^d \setminus \{0\}$ and any $r > 0$, we define

$$H(y, r) := \big\{z \in B(y, r): \ z \cdot y \geqslant 0\big\}.$$

Lemma 5.1. *Let $T > 0$. There exist constants $\varepsilon = \varepsilon(d, \alpha, T) > 0$, $L_2 = L_2(d, \alpha, T) \geqslant 3$ such that the following holds: for all $t \geqslant T$, $R > 0$, x and y satisfying $|x| > L_2 R$, $|y| > R$ and $y \in B(x, 9\sqrt{t})$,*

$$\mathbb{P}_x\big(X_t^{1,\overline{B}^c_R} \in H(y, \sqrt{t}/2)\big) \geqslant \varepsilon.$$

Proof. It follows from Theorem 2.1 that there exist constants $c_1 > 0$ and $c_2 > 1$ such that

$$\begin{aligned}
\mathbb{P}_x\big(X_t^1 \in H(y, \sqrt{t}/2)\big) &\geqslant \inf_{w \in B(y, 9\sqrt{t})} \mathbb{P}_w\big(X_t^1 \in H(y, \sqrt{t}/2)\big) \\
&\geqslant c_1 \inf_{w \in B(y, 9\sqrt{t})} \int\limits_{H(y, \frac{1}{2}\sqrt{t})} \widetilde{\Psi}_{d,\alpha,1,c_2^{-1}}(t, w, z)\, dz.
\end{aligned}$$

If $T < 1$ and $t \in [T, 1]$, then clearly

$$\begin{aligned}
\inf_{w \in B(y, 9\sqrt{t})} \int\limits_{H(y, \frac{1}{2}\sqrt{t})} \widetilde{\Psi}_{d,\alpha,1,c_2^{-1}}(t, w, z)\, dz &\geqslant c_3 \inf_{w \in B(y, 9)} \int\limits_{H(y, \frac{T}{2})} \left(1 \wedge \frac{1}{|w - z|^{d+\alpha}}\right) dz \\
&\geqslant c_4 10^{-d-\alpha}\big|B(0, 1/2)\big|.
\end{aligned}$$

If $t > 1$, then

$$\begin{aligned}
&\inf_{w \in B(y, 9\sqrt{t})} \int\limits_{H(y, \frac{1}{2}\sqrt{t})} \widetilde{\Psi}_{d,\alpha,1,c_2^{-1}}(t, w, z)\, dz \\
&\quad\geqslant \inf_{w \in B(y, 9\sqrt{t})} \int\limits_{H(y, \frac{1}{2}\sqrt{t})} t^{-d/2} \exp\left(-c_2 \frac{|z - w|^2}{t}\right) dz \\
&\quad\geqslant c_5 \inf_{w \in B(y, 9)} \int\limits_{H(y, \frac{1}{2})} \exp\big(-c_2|z - w|^2\big)\, dz \geqslant c_6 e^{-c_2 10^2}\big|B(0, 1/2)\big|.
\end{aligned}$$

Hence there is $\varepsilon \in (0, 1/4)$ so that for any $t \geqslant T$, $x \in \mathbb{R}^d$ and $y \in B(x, 9\sqrt{t})$,

$$\varepsilon < \frac{1}{2}\mathbb{P}_x\big(X_t^1 \in H(y, \sqrt{t}/2)\big). \tag{5.1}$$

Since $d \geqslant 3$, we may choose $L_2 \geqslant 3$ so that $C_2(L_2^{2-d} + L_2^{\alpha-d}) \leqslant \varepsilon$. By Lemma 4.1, for all x with $|x| > L_2 R$, we have

$$\begin{aligned}\mathbb{P}_x\big(\tau^1_{\overline{B}^c_R} \leqslant t\big) &\leqslant \mathbb{P}_x\big(T^1_{B(0,R)} < \infty\big) \leqslant C_2\frac{R^d}{R^2 + R^\alpha}\big(|x|^{2-d} + |x|^{\alpha-d}\big)\\ &\leqslant C_2\bigg(\frac{R^2}{R^2+R^\alpha}L_2^{2-d} + \frac{R^\alpha}{R^2+R^\alpha}L_2^{\alpha-d}\bigg)\\ &\leqslant C_2\big(L_2^{2-d} + L_2^{\alpha-d}\big) \leqslant \varepsilon.\end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2) gives

$$\begin{aligned}\mathbb{P}_x\big(X_t^{1,\overline{B}^c_R} \in H(y, \sqrt{t}/2)\big) &= \mathbb{P}_x\big(\tau^1_{\overline{B}^c_R} > t\big) - \mathbb{P}_x\big(X_t^{1,\overline{B}^c_R} \notin H(y, \sqrt{t}/2); \tau^1_{\overline{B}^c_R} > t\big)\\ &\geqslant \mathbb{P}_x\big(\tau^1_{\overline{B}^c_R} > t\big) - \mathbb{P}_x\big(X_t^1 \notin H(y, \sqrt{t}/2)\big)\\ &\geqslant (1-\varepsilon) - (1-2\varepsilon) = \varepsilon.\end{aligned}$$

This proves the lemma. □

Lemma 5.2. *Let $T > 0$ and $L_2 = L_2(d, \alpha, T/8)$ be the constant in Lemma* 5.1. *There exists a constant $c = c(\alpha, d, T) > 0$ such that for all $m > 0$, $R > 0$, $t \geqslant T/m$ and x, y satisfying $|x| > L_2 R$, $|y| > L_2 R$, $|x - y| \leqslant m^{1/2-1/\alpha}\sqrt{t}/6$, we have*

$$p^m_{\overline{B}^c_R}(t, x, y) \geqslant c m^{d/\alpha - d/2} t^{-d/2}.$$

Proof. We first prove the lemma for $m = 1$. Assume without loss of generality that $|y| \geqslant |x|$. If $\delta_{\overline{B}^c_R}(y) > \sqrt{t}/2$, then $\delta_{\overline{B}^c_R}(x) \geqslant \delta_{\overline{B}^c_R}(y) - |x - y| \geqslant \sqrt{t}/3$, and hence the lemma follows immediately from Theorem 3.8.

Now assume $\delta_{\overline{B}^c_R}(y) \leqslant \sqrt{t}/2$. By the semigroup property and Theorem 2.2 we have

$$\begin{aligned}p^1_{\overline{B}^c_R}(t, x, y) &\geqslant \int\limits_{H(y,(t/2)^{1/2})} p^1_{\overline{B}^c_R}(t/2, x, z)\, p^1_{\overline{B}^c_R}(t/2, z, y)\, dz\\ &\geqslant c_1 \mathbb{P}_x\big(X_{t/2}^{1,\overline{B}^c_R} \in H\big(y, (t/2)^{1/2}\big)\big)\, p^1_{\overline{B}^c_R}\big((t/2) - \delta_{\overline{B}^c_R}(y)^2, y, y\big).\end{aligned} \tag{5.3}$$

By Lemma 5.1 we have $\mathbb{P}_x(X_{t/2}^{1,\overline{B}^c_R} \in H(y, (t/2)^{1/2})) \geqslant \varepsilon$.

Note that $t \geqslant s := (t/2) - \delta_{\overline{B}^c_R}(y)^2 \geqslant t/4 \geqslant T/4$. Hence by the semigroup property, the Cauchy–Schwarz inequality and Lemma 5.1

$$p^1_{\overline{B}^c_R}(s, y, y) \geqslant \int\limits_{H(y,\sqrt{s}/2)} \left(p^1_{\overline{B}^c_R}(s/2, y, z)\right)^2 dz$$

$$\geqslant \frac{2}{|B(y, \sqrt{s}/2)|} \mathbb{P}_y\left(X^{1,\overline{B}^c_R}_{s/2} \in H(y, \sqrt{s}/2)\right)^2 \geqslant c_2 s^{-d/2} \geqslant c_2 t^{-d/2}.$$

Thus by (5.3) we have $p^1_{\overline{B}^c_R}(t, x, y) \geqslant c_3 t^{-d/2}$.

Now we consider the general case $m > 0$, $|x - y| \leqslant m^{1/2-1/\alpha}\sqrt{t}/6$, and $t \geqslant T/m$. We apply (2.7) to the previous case and get

$$p^m_{\overline{B}^c_R}(t, x, y) = m^{d/\alpha} p^1_{B(0,m^{1/\alpha}R)^c}\left(mt, m^{1/\alpha}x, m^{1/\alpha}y\right) \geqslant c_3 m^{d/\alpha-d/2} t^{-d/2}. \qquad \square$$

Proposition 5.3. *Let $T > 0$ and $L_2 = L_2(d, \alpha, T/(16))$ be the constant in Lemma 5.1. There exist constants $c_1 = c_1(\alpha, d, T) > 0$ and $C_3 = C_3(\alpha, d, T) > 0$ such that for all $R, m > 0$, $t \geqslant T/m$, and x, y satisfying $|x| > L_2R$, $|y| > L_2R$,*

$$p^m_{\overline{B}^c_R}(t, x, y) \geqslant c_1 m^{d/\alpha-d/2} t^{-d/2} \exp\left(-C_3\left(m^{1/\alpha}|x - y| \wedge m^{2/\alpha-1}\frac{|x - y|^2}{t}\right)\right).$$

Proof. By Lemma 5.2, we only need to prove the proposition for $|x-y| > \frac{1}{6}m^{1/2-1/\alpha}\sqrt{t}$, which we will assume throughout the proof.

We first prove the lemma for $m = 1$. If $t < (60R)^2$, then $\delta_{\overline{B}^c_R}(x) \geqslant (L_2 - 1)R \geqslant 2R > (30)^{-1}t^{1/2}$. Thus, in this case, the lemma follows immediately from Theorem 3.8.

Suppose $t \geqslant T \vee (60R)^2$ and $|x - y| \leqslant \frac{1}{6}\sqrt{t}$. As one of $|x|$ and $|y|$ must be no less than $|x-y|/2$, we assume without loss of generality that $|y| \geqslant |x-y|/2$. Let $x_0 := x + 20^{-1}\sqrt{t}x/|x|$ and observe that $B(x_0, 20^{-1}\sqrt{t}) \subset \overline{B}^c_{|x|} \subset \overline{B}^c_R$.

Since $|x - y| > \frac{1}{6}\sqrt{t}$, we get for every $z \in B(x_0, 20^{-1}\sqrt{t})$,

$$|x - z| \leqslant |x_0 - z| + \frac{1}{20}\sqrt{t} \leqslant \frac{1}{10}\sqrt{t} \leqslant \frac{1}{6}\sqrt{t/2}$$

and

$$|z - y| \leqslant |x - y| + |x - z| \leqslant |x - y| + \frac{1}{10}\sqrt{t} \leqslant 2|x - y|.$$

Moreover, since $R < \frac{1}{60}\sqrt{t}$,

$$\delta_{\overline{B}^c_R}(y) = |y| - R \geqslant \frac{1}{2}|x - y| - \frac{1}{60}\sqrt{t} \geqslant \frac{1}{12}\sqrt{t} - \frac{1}{60}\sqrt{t} = \frac{1}{15}\sqrt{t}$$

and, for $z \in B(x_0, \frac{1}{60}\sqrt{t})$,

$$\delta_{\overline{B}^c_R}(z) = |z| - R \geqslant |x_0| - |x_0 - z| - \frac{1}{60}\sqrt{t} \geqslant |x| + \frac{1}{20}\sqrt{t} - \frac{1}{60}\sqrt{t} - \frac{1}{60}\sqrt{t} \geqslant \frac{1}{60}\sqrt{t}.$$

Thus by the semigroup property, Theorem 3.8 and Lemma 5.2, there exist positive constants $c_i = c_i(d, \alpha, T)$, $i = 1, \ldots, 3$, such that

$$\begin{aligned}
p^1_{\overline{B}^c_R}(t,x,y) &= \int_{\overline{B}^c_R} p^1_{\overline{B}^c_R}(t/2,x,z)p^1_{\overline{B}^c_R}(t/2,z,y)\,dz \\
&\geqslant \int_{B(x_0,\frac{1}{60}\sqrt{t})} p^1_{\overline{B}^c_R}(t/2,x,z)p^1_{\overline{B}^c_R}(t/2,z,y)\,dz \\
&\geqslant c_1 \int_{B(x_0,\frac{1}{60}\sqrt{t})} (t/2)^{-d/2}(t/2)^{-d/2}\exp\left(-c_2\left(|z-y|\wedge\frac{|z-y|^2}{t/2}\right)\right)dz \\
&\geqslant c_3 t^{-d/2}\exp\left(-4c_2\left(|x-y|\wedge\frac{|x-y|^2}{t}\right)\right).
\end{aligned}$$

Now we consider the general case $m>0$ and $t\geqslant T/m$. We apply (2.7) to the previous case and get

$$\begin{aligned}
p^m_{\overline{B}^c_R}(t,x,y) &= m^{d/\alpha}p^1_{B(0,m^{1/\alpha}R)^c}\big(mt,m^{1/\alpha}x,m^{1/\alpha}y\big) \\
&\geqslant c_3 m^{d/\alpha-d/2}t^{-d/2}\exp\left(-C_3\left(m^{1/\alpha}|x-y|\wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\right)\right). \qquad \square
\end{aligned}$$

Theorem 5.4. *Suppose that M, T, R are positive constants, and that D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (r_0,Λ_0) and $D^c\subset B(0,R)$. There exist positive constants $c_i=c_i(d,\alpha,r_0,\Lambda_0,R,M,T)$, $i=1,2$, such that for all $0<m\leqslant M$ and $(t,x,y)\in[T/m,\infty)\times D\times D$,*

$$\begin{aligned}
p^m_D(t,x,y) \geqslant c_1\left(1\wedge\frac{\delta_D(x)}{1\wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1\wedge\frac{\delta_D(y)}{1\wedge t^{1/\alpha}}\right)^{\alpha/2} m^{d/\alpha-d/2}t^{-d/2} \\
\cdot\exp\left(-4c_2\left(m^{1/\alpha}|x-y|\wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\right)\right).
\end{aligned}$$

Proof. Without loss of generality, we assume $M=T=1$. By Theorem 4.6, we may assume $t>3/m$. For x and y in D, let $v\in\mathbb{R}^d$ be any unit vector satisfying $x\cdot v\geqslant 0$ and $y\cdot v\geqslant 0$. Recall that C_1 is the constant in Theorem 2.1 and that $C_3=C_3(\alpha,d,1)$ is the constant in Proposition 5.3. Let $A:=4\vee(C_1C_3)$ and $L_3:=L_2(d,\alpha,(16)^{-1})\vee L_2(d,\alpha,(16)^{-1}A^{-\alpha})$, where L_2 is given by Proposition 5.3. Define

$$x_0:=x+m^{-1/\alpha}L_3^2Rv \quad\text{and}\quad y_0:=y+m^{-1/\alpha}L_3^2Rv.$$

Then

$$|x_0|^2=|x|^2+m^{-2/\alpha}\big(L_3^2R\big)^2+2m^{-1/\alpha}L_3^2Rx\cdot v\geqslant\big(m^{-1/\alpha}L_3^2R\big)^2\geqslant\big(L_3^2R\big)^2,$$

and similarly, $|y_0|^2\geqslant(L_3^2R)^2$. By the semigroup property and Theorem 4.6, for every $t\in(3/m,\infty)$,

$$
\begin{aligned}
p_D^m(t,x,y) &= \int_D \int_D p_D^m(1/m,x,z)\,p_D^m(t-2/m,z,w)\,p_D^m(1/m,w,y)\,dz\,dw \\
&\geqslant c_1\big(1\wedge\delta_D(x)\big)^{\alpha/2}\big(1\wedge\delta_D(y)\big)^{\alpha/2} f_1(t,x,y),
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
f_1(t,x,y) = \int_{D\times D} &\big(1\wedge\delta_D(z)\big)^{\alpha/2}\left(m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|x-z|)}{|x-z|^{d+\alpha}}\right) p_D^m(t-2/m,z,w) \\
&\cdot \big(1\wedge\delta_D(w)\big)^{\alpha/2}\left(m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|w-y|)}{|w-y|^{d+\alpha}}\right) dz\,dw.
\end{aligned}
$$

By Proposition 5.3, for $z, w \in B(0, L_3R)^c$ and $t \in (3/m, \infty)$ (note that $t-2 \geqslant 1/m$),

$$
\begin{aligned}
&p_D^m(t-2/m,z,w) \\
&\quad\geqslant p_{B_R^c}^m(t-2/m,z,w) \\
&\quad\geqslant c_2 m^{d/\alpha-d/2}(t-2/m)^{-d/2}\exp\left(-C_3\left(m^{1/\alpha}|z-w|\wedge m^{2/\alpha-1}\frac{|z-w|^2}{t-2/m}\right)\right).
\end{aligned}
\tag{5.5}
$$

Moreover, since

$$
m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}r)}{r^{d+\alpha}} = m^{d/\alpha}\left(1\wedge\frac{\phi(4m^{1/\alpha}r)}{(m^{1/\alpha}r)^{d+\alpha}}\right),
$$

we have by Lemma 4.5,

$$
m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|x-z|)}{|x-z|^{d+\alpha}} \geqslant c_3\left(m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|x_0-z|)}{|x_0-z|^{d+\alpha}}\right)
\tag{5.6}
$$

and

$$
m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|w-y|)}{|w-y|^{d+\alpha}} \geqslant c_3\left(m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|w-y_0|)}{|w-y_0|^{d+\alpha}}\right).
\tag{5.7}
$$

Since $m \leqslant 1$, the upper bound estimate in Theorem 1.1(i) and (5.5)–(5.7) imply that

$$
\begin{aligned}
f_1(t,x,y) \geqslant c_2 \int_{B(0,L_3R)^c\times B(0,L_3R)^c} &\big(1\wedge\delta_D(z)\big)^{\alpha/2}\left(m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|x-z|)}{|x-z|^{d+\alpha}}\right) m^{d/\alpha-d/2} \\
&\cdot (t-2/m)^{-d/2}\exp\left(-C_3\left(m^{1/\alpha}|z-w|\wedge m^{2/\alpha-1}\frac{|z-w|^2}{t-2/m}\right)\right)\big(1\wedge\delta_D(w)\big)^{\alpha/2} \\
&\cdot \left(m^{d/\alpha}\wedge m^{-1}\frac{\phi(4m^{1/\alpha}|w-y|)}{|w-y|^{d+\alpha}}\right) dz\,dw
\end{aligned}
$$

$$\begin{aligned}
&\geqslant c_4 \int\limits_{B(0,L_3R)^c\times B(0,L_3R)^c} \left(m^{d/\alpha} \wedge m^{-1}\frac{\phi(4m^{1/\alpha}|x_0-z|)}{|x_0-z|^{d+\alpha}}\right) m^{d/\alpha-d/2} \\
&\quad \cdot (t-2/m)^{-d/2}\exp\left(-C_3\left(m^{1/\alpha}|z-w| \wedge m^{2/\alpha-1}\frac{|z-w|^2}{t-2/m}\right)\right) \\
&\quad \cdot \left(m^{d/\alpha} \wedge m^{-1}\frac{\phi(4m^{1/\alpha}|w-y_0|)}{|w-y_0|^{d+\alpha}}\right) dz\,dw.
\end{aligned}$$

Recall $A = 4 \vee (C_1C_3)$. By the change of variables $\widehat{z} = Az$, $\widehat{w} = Aw$, and Theorem 2.1, we have that

$$\begin{aligned}
f_1(t,x,y) &\geqslant c_5 \int\limits_{B(0,AL_3R)^c\times B(0,AL_3R)^c} \left(m^{d/\alpha} \wedge m^{-1}\frac{\phi(4A^{-1}m^{1/\alpha}|Ax_0-\widehat{z}|)}{|Ax_0-\widehat{z}|^{d+\alpha}}\right) m^{d/\alpha-d/2} \\
&\quad \cdot (t-2/m)^{-d/2}\exp\left(-C_1^{-1}\left(m^{1/\alpha}|\widehat{z}-\widehat{w}| \wedge m^{2/\alpha-1}\frac{|\widehat{z}-\widehat{w}|^2}{t-2/m}\right)\right) \\
&\quad \cdot \left(m^{d/\alpha} \wedge m^{-1}\frac{\phi(4A^{-1}m^{1/\alpha}|\widehat{w}-Ay_0|)}{|\widehat{w}-Ay_0|^{d+\alpha}}\right) d\widehat{z}\,d\widehat{w} \\
&\geqslant c_5 \int\limits_{B(0,AL_3R)^c\times B(0,AL_3R)^c} \left(m^{d/\alpha} \wedge m^{-1}\frac{\phi(m^{1/\alpha}|Ax_0-\widehat{z}|)}{|Ax_0-\widehat{z}|^{d+\alpha}}\right) m^{d/\alpha-d/2} \\
&\quad \cdot (t-2/m)^{-d/2}\exp\left(-C_1^{-1}\left(m^{1/\alpha}|\widehat{z}-\widehat{w}| \wedge m^{2/\alpha-1}\frac{|\widehat{z}-\widehat{w}|^2}{t-2/m}\right)\right) \\
&\quad \cdot \left(m^{d/\alpha} \wedge m^{-1}\frac{\phi(m^{1/\alpha}|\widehat{w}-Ay_0|)}{|\widehat{w}-Ay_0|^{d+\alpha}}\right) d\widehat{z}\,d\widehat{w} \\
&\geqslant c_6 \int\limits_{B(0,AL_3R)^c\times B(0,AL_3R)^c} p^m(1/m, Ax_0, \widehat{z})\, p^m(t-2/m, \widehat{z}, \widehat{w}) \\
&\quad \cdot p^m(1/m, \widehat{w}, Ay_0)\, dz\,dw \\
&\geqslant c_6 \int\limits_{B(0,AL_3R)^c\times B(0,AL_3R)^c} p^m_{B(0,AL_3R)^c}(1/m, Ax_0, \widehat{z})\, p^m_{B(0,AL_3R)^c}(t-2/m, \widehat{z}, \widehat{w}) \\
&\quad \cdot p^m_{B(0,AL_3R)^c}(1/m, \widehat{w}, Ay_0)\, dz\,dw \\
&= c_6\, p^m_{B(0,AL_3R)^c}(t, Ax_0, Ay_0).
\end{aligned}$$

Now using (2.7) and Proposition 5.3 again (recall that $|x_0|, |y_0| \geqslant L_3(L_3R)$), we conclude that

$$\begin{aligned}
&p^m_{B(0,AL_3R)^c}(t, Ax_0, Ay_0) \\
&\quad = A^{-d}\, p^{mA^\alpha}_{B(0,L_3R)^c}\left(A^{-\alpha}t, x_0, y_0\right)
\end{aligned}$$

$$\geqslant c_7 m^{d/\alpha-d/2} t^{-d/2} \exp\left(-c_8\left(A m^{1/\alpha}|x-y| \wedge A^2 m^{2/\alpha-1}\frac{|x_0-y_0|^2}{t}\right)\right)$$
$$\geqslant c_7 m^{d/\alpha-d/2} t^{-d/2} \exp\left(-c_9\left(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t}\right)\right).$$

Combining the last two displays with (4.10) completes the proof. □

Proof of the lower bound estimate in Theorem 1.2. The lower bound estimate in Theorem 1.2 now follows from Theorems 1.1(i), 4.6 and 5.4. This completes the proof of Theorem 1.2. □

6. Green function estimate

In this section, we present a proof of Theorem 1.3.

Proof of Theorem 1.3. In view of the scaling property (2.8) of G_D^m, we may and do assume that $M = 1/2$. Throughout this proof, $m \in (0, 1/2]$. It follows from Theorem 1.2 that there exists $c_i > 1$, $i = 1, 2$, such that for every $m \in (0, 1/2]$, $t > 0$ and $(x, y) \in D \times D$,

$$p_D^m(t,x,y) \leqslant c_1\left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,1,c_2}(t,x,y)$$

and

$$p_D^m(t,x,y) \geqslant c_1^{-1}\left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,1,1/c_2}(t,x,y).$$

For any $c > 0$, define

$$J_c := \int_0^\infty \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,1,c}(t,x,y)\,dt.$$

Then it suffices to show that

$$J_c \asymp \frac{1+(m^{1/\alpha}|x-y|)^{2-\alpha}}{|x-y|^{d-\alpha}}\left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1}\right)^{\alpha/2}.$$

Without loss of generality, we will assume $c = 1$ and denote J_1 simply by J.

Using a change of variable, we see that

$$J = I_1 + \left(1 \wedge \delta_D(x)\right)^{\alpha/2}\left(1 \wedge \delta_D(y)\right)^{\alpha/2}\left(I_2\left(|x-y|\right) + m^{d/\alpha-1} I_3\left(m^{1/\alpha}|x-y|\right)\right)$$

where

$$I_1 := \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2}\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right)dt,$$

$$I_2(r) := \int_1^{1/m} t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha} r)}{r^{d+\alpha}}\, dt \quad \text{and} \quad I_3(r) := \int_1^{\infty} t^{-d/2} e^{-r \wedge (r^2/t)}\, dt.$$

Note that for every $a \in [0, \infty)$

$$\int_a^{\infty} t^{-d/2} e^{-r^2/t}\, dt = r^{2-d} \int_0^{r^2/a} u^{d/2-2} e^{-u}\, du.$$

Thus for $r \in (0, 1]$

$$I_3(r) = r^{2-d} \int_0^{r^2} u^{d/2-2} e^{-u}\, du \asymp r^{2-d} \int_0^{r^2} u^{d/2-2}\, du = \frac{2}{d-2},$$

while for $r > 1$,

$$\begin{aligned} r^{2-d} \int_0^1 u^{d/2-2} e^{-u}\, du &\leqslant \int_r^{\infty} t^{-d/2} e^{-r^2/t}\, dt \leqslant I_3(r) = \int_1^r t^{-d/2} e^{-r}\, dt + \int_r^{\infty} t^{-d/2} e^{-r^2/t}\, dt \\ &\leqslant c_3 e^{-r} + r^{2-d} \int_0^r u^{d/2-2} e^{-u}\, du \leqslant c_4 r^{2-d}. \end{aligned}$$

Thus we have

$$I_3(r) \asymp 1 \wedge r^{2-d}. \tag{6.1}$$

Noting that $m \in (0, 1/2]$, so when $m^{1/\alpha} r \leqslant 1$,

$$\begin{aligned} I_2(r) &\asymp \int_1^{1/m} \left(t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right) dt = \int_1^{r^\alpha \vee 1} \frac{t}{r^{d+\alpha}}\, dt + \int_{r^\alpha \vee 1}^{1/m} t^{-d/\alpha}\, dt \\ &= \frac{1}{2r^{d+\alpha}} \left(\left(r^\alpha \vee 1\right)^2 - 1 \right) + \frac{\alpha}{d-\alpha} \left(\left(r^\alpha \vee 1\right)^{1-d/\alpha} - m^{d/\alpha - 1} \right) \\ &\asymp 1 \wedge r^{\alpha - d}. \end{aligned} \tag{6.2}$$

If $m^{1/\alpha} r > 1$, then

$$I_2(r) \asymp \int_1^{1/m} \left(t^{-d/\alpha} \wedge \frac{t e^{-m^{1/\alpha} r}}{r^{d+\alpha}} \right) dt$$

and a change of variable $s = tm$ gives

$$I_2(r) \asymp m^{d/\alpha-1} \int_m^1 \left(s^{-d/\alpha} \wedge \frac{s e^{-m^{1/\alpha} r}}{(m^{1/\alpha} r)^{d+\alpha}} \right) ds = m^{d/\alpha-1} \int_m^1 \frac{s e^{-m^{1/\alpha} r}}{(m^{1/\alpha} r)^{d+\alpha}} \, ds$$
$$\asymp m^{d/\alpha-1} \frac{e^{-m^{1/\alpha} r}}{(m^{1/\alpha} r)^{d+\alpha}} = \frac{e^{-m^{1/\alpha} r}}{(m^{1/\alpha} r)^2 r^{d-\alpha}}. \tag{6.3}$$

(i) Suppose $m^{1/\alpha}|x-y| \leqslant 1$. Since $\phi(m^{1/\alpha}|x-y|) \asymp 1$, it follows from [8, (4.3), (4.4) and (4.6)] that

$$I_1 \asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2}.$$

Thus, we have by (6.1) and (6.2) that

$$J \asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|} \right)^{\alpha/2}$$
$$+ \big(1 \wedge \delta_D(x)\big)^{\alpha/2} \big(1 \wedge \delta_D(y)\big)^{\alpha/2} \left(1 \wedge \frac{1}{|x-y|^{d-\alpha}} + m^{d/\alpha-1} \right)$$
$$\asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1} \right)^{\alpha/2}. \tag{6.4}$$

We have arrived the last display above by considering the cases $|x-y| \geqslant 1$ and $|x-y| < 1$ separately.

(ii) Suppose $m^{1/\alpha}|x-y| > 1$. For $0 \leqslant t \leqslant 1$, we have $0 \leqslant mt \leqslant 1/2$ and so

$$\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \right) = m^{d/\alpha} \left((mt)^{-d/\alpha} \wedge \frac{(mt)\phi(m^{1/\alpha}|x-y|)}{(m^{1/\alpha}|x-y|)^{d+\alpha}} \right)$$
$$= \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}.$$

Thus by the change of variable $u = \frac{|x-y|^\alpha}{t}$, we have

$$I_1 = \frac{\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \int_0^1 t \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}} \right)^{\alpha/2} dt$$
$$= \frac{\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d-\alpha}} \int_{|x-y|^\alpha}^{\infty} u^{-3} \left(1 \wedge \frac{\sqrt{u}\,\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) \left(1 \wedge \frac{\sqrt{u}\,\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}} \right) du. \tag{6.5}$$

Note that since $|x-y| \geqslant m^{-1/\alpha} > 2^{1/\alpha}$,

$$\begin{aligned}
&\int_{|x-y|^\alpha}^{\infty} u^{-3}\left(1\wedge \frac{\sqrt{u}\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge \frac{\sqrt{u}\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) du \\
&\quad= \int_{|x-y|^\alpha}^{\infty} u^{-2}\left(u^{-1/2}\wedge \frac{\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(u^{-1/2}\wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) du \\
&\quad\leqslant \int_{|x-y|^\alpha}^{\infty} u^{-2}\, du \left(1\wedge \frac{\delta_D(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge \frac{\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \\
&\quad= |x-y|^{-\alpha}\left(1\wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha/2}\left(1\wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha/2}.
\end{aligned}$$

Thus it follows from (6.5) that

$$\begin{aligned}
I_1 &\leqslant \frac{\phi(1)}{|x-y|^d}\left(1\wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha/2}\left(1\wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha/2} \\
&\leqslant \frac{\phi(1)m^{2/\alpha}}{|x-y|^{d-2}}\big(1\wedge \delta_D(x)\big)^{\alpha/2}\big(1\wedge \delta_D(y)\big)^{\alpha/2},
\end{aligned} \tag{6.6}$$

where in the last inequality, we used the assumption $m^{1/\alpha}|x-y| \geqslant 1$. Recalling $m \in (0, 1/2]$, we thus have by (6.1), (6.3) and (6.6) that

$$\begin{aligned}
J &\asymp \big(1\wedge \delta_D(x)\big)^{\alpha/2}\big(1\wedge \delta_D(y)\big)^{\alpha/2}\left(\frac{e^{-m^{1/\alpha}|x-y|}}{(m^{1/\alpha}|x-y|)^2|x-y|^{d-\alpha}} + \frac{m^{2/\alpha-1}}{|x-y|^{d-2}}\right) \\
&= \big(1\wedge \delta_D(x)\big)^{\alpha/2}\big(1\wedge \delta_D(y)\big)^{\alpha/2}\left(\frac{e^{-m^{1/\alpha}|x-y|}}{(m^{1/\alpha}|x-y|)^2|x-y|^{d-\alpha}} + \frac{(m^{1/\alpha}|x-y|)^{2-\alpha}}{|x-y|^{d-\alpha}}\right) \\
&\asymp \big(1\wedge \delta_D(x)\big)^{\alpha/2}\big(1\wedge \delta_D(y)\big)^{\alpha/2}\frac{(m^{1/\alpha}|x-y|)^{2-\alpha}}{|x-y|^{d-\alpha}} \\
&\asymp \frac{1+(m^{1/\alpha}|x-y|)^{2-\alpha}}{|x-y|^{d-\alpha}}\left(1\wedge \frac{\delta_D(x)}{|x-y|\wedge 1}\right)^{\alpha/2}\left(1\wedge \frac{\delta_D(y)}{|x-y|\wedge 1}\right)^{\alpha/2}.
\end{aligned}$$

This combing with (6.4) completes the proof of the theorem. □

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